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On a Spherical Vortex

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VI. *On a Spherical Vortex.*

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1. IN a paper published by the author in the 'Philosophical Transactions' for 1884, "On the Motion of Fluid, part of which is moving rotationally and part irrotationally," a certain case of motion, symmetrical with regard to an axis, was noticed (see pp. 403–405).

Taking the axis of symmetry as axis of z, and the distance of any point from it as r, and allowing for a difference of notation, it was shown that the surfaces

$$r^2\left(\frac{r^2}{a^2}+\frac{(z-Z)^2}{c^2}-1\right)=\text{constant},$$

where a, c are fixed constants, and Z any arbitrary function of the time, always contain the same particles of fluid in a possible case of motion.

The surfaces are of invariable form. If the constant be less than $-\frac{1}{4}a^2$, the surfaces are imaginary; if the constant lie between $-\frac{1}{4}a^2$ and zero they are ring-shaped; if the constant be zero, the single surface represented breaks up into an evanescent cylinder and an ellipsoid of revolution; if the constant be positive, the surfaces have the axis of revolution for an asymptote.

The velocity perpendicular to the axis of symmetry is

$$2\frac{k}{c^2}r(z-Z);$$

the velocity parallel to the axis of symmetry is

$$\dot{Z}-\frac{2k}{a^2}(2r^2-a^2)-2\frac{k}{c^2}(z-Z)^2;$$

where k is a fixed constant and $\dot{Z}=dZ/dt$.

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These expressions (which make the velocity infinitely great at infinity) cannot apply to a possible case of fluid motion extending to infinity. Hence the fluid moving in the above manner must be limited by a surface of finite dimensions. This limiting surface must always contain the same particles of fluid.

Where, as in the present case, the surfaces containing the same particles of fluid are of invariable form, it is possible to imagine the fluid limited by any one of them, provided a rigid frictionless boundary having the shape of the limiting surface be supplied, and the boundary be supposed to move parallel to the axis of z with velocity $\dot{Z}$. Then the above expressions give the velocity components of a possible rotational motion inside the boundary. So much was pointed out in the paper cited above.

2. But a case of much greater interest is obtained when it is possible to limit the fluid moving in the above manner by one of the surfaces containing always the same particles of fluid, and to discover either an irrotational or rotational motion filling all space external to the limiting surface which is continuous with the motion inside it as regards velocity normal to the limiting surface and pressure.

3. It is the object of this paper to discuss such a case, the motion found external to the limiting surface being an irrotational motion, and the tangential velocity at the limiting surface, as well as the normal velocity, and the pressure being continuous.

The particular surface (containing the same particles) which is selected is obtained by supposing that the constant vanishes, and also that $c = a$. Then this surface breaks up into the evanescent cylinder

$$r^2 = 0,$$

and the sphere

$$r^2 + (z - Z)^2 = a^2.$$

The molecular rotation is given by $\omega = 5kr/a^2$, so that the molecular rotation along the axis vanishes, and therefore the vortex sphere still possesses to some extent the character of a vortex ring.

The irrotational motion outside a sphere moving in a straight line is known, and it is shown in this paper that it will be continuous with the rotational motion inside the sphere provided a certain relation be satisfied.

This relation may be expressed thus :—

The cyclic constant of the spherical vortex is five times the product of the radius of the sphere and the uniform velocity with which the vortex sphere moves along its axis.

The analytic expression of the same relation is

$$4k = 3\dot{Z}.$$

This makes

$$\omega = 15\dot{Z}r/(4a^2).$$

All the particulars of the motion are placed together in the Table below, in which the notation employed is as follows :—

If the velocity parallel to the axis of r be τ, and the velocity parallel to the axis of z be w, then the molecular rotation is given by

$$2\omega = \frac{\partial \tau}{\partial z} - \frac{\partial w}{\partial r}.$$

Also p is the pressure, ρ the density, and V the potential of the impressed forces. The minimum value of $p/\rho + \mathrm{V}$ is Π/ρ, where Π/ρ must be determined from the initial conditions.

Further R, θ are such that

$$r = \mathrm{R} \sin \theta,$$
$$z - \mathrm{Z} = \mathrm{R} \cos \theta.$$

The whole motion depends on the following constants :—

(1.) The radius of the sphere, a.

(2.) The uniform velocity with which the vortex sphere moves along its axis, $\dot{\mathrm{Z}}$.

(3.) The minimum value of $p/\rho + \mathrm{V}$, viz., Π/ρ.

	Rotational motion inside sphere.	At the surface of the sphere.	Irrotational motion outside sphere.
Velocity parallel to axis of r	$3\dot{\mathrm{Z}}r(z - \mathrm{Z})/(2a^2)$	$\frac{3}{2}\dot{\mathrm{Z}} \sin \theta \cos \theta$	$3a^3\dot{\mathrm{Z}}r(z - \mathrm{Z})/(2\mathrm{R}^5)$
Velocity parallel to axis of z	$\dot{\mathrm{Z}}\{5a^2 - 3(z - \mathrm{Z})^2 - 6r^2\}/(2a^2)$	$\dot{\mathrm{Z}}(1 - \frac{3}{2}\sin^2\theta)$	$a^3\dot{\mathrm{Z}}\{3(z - \mathrm{Z})^2 - \mathrm{R}^2\}/(2\mathrm{R}^5)$
$p/\rho + \mathrm{V} - \Pi/\rho$	$\frac{9\dot{\mathrm{Z}}^2}{8a^4}[(r^2 - \frac{1}{2}a^2)^2 - \{(z - \mathrm{Z})^2 - a^2\}^2 + a^4]$	$\frac{9}{8}\dot{\mathrm{Z}}^2\cos^2\theta + \frac{9}{32}\dot{\mathrm{Z}}^2$	$\frac{1}{8}\dot{\mathrm{Z}}^2\left[\begin{array}{l}\frac{9}{4} + \{5 - 4(a/\mathrm{R})^3 - (a/\mathrm{R})^6\} \\ + 3\cos^2\theta\{4(a/\mathrm{R})^3 - (a/\mathrm{R})^6\}\end{array}\right]$
Current function	$3\dot{\mathrm{Z}}r^2\{\mathrm{R}^2 - \frac{5}{3}a^2\}/(4a^2)$	. . . .	$-a^3\dot{\mathrm{Z}}r^2/(2\mathrm{R}^3)$
Surfaces containing the same particles of fluid throughout the motion	$3\dot{\mathrm{Z}}r^2\{\mathrm{R}^2 - a^2\}/(4a^2) = \text{constant}$	. . . .	$\dot{\mathrm{Z}}r^2(\mathrm{R}^3 - a^3)/(2\mathrm{R}^3) = \text{constant}$
Velocity potential	. . . .	. . . .	$-a^3\dot{\mathrm{Z}}(z - \mathrm{Z})/(2\mathrm{R}^3)$
Molecular rotation	$15\dot{\mathrm{Z}}r/(4a^2)$	. . . .	. . . .
Cyclic constant of vortex	$5a\dot{\mathrm{Z}}$	. . . .	. . . .
Kinetic energy	$23\pi\rho a^3\dot{\mathrm{Z}}^2/21$	. . . .	$\pi\rho a^3\dot{\mathrm{Z}}^2/3$

4. If c be not equal to a, then the surface containing the same particles, when the constant vanishes, breaks up into an evanescent cylinder and an ellipsoid of revolution.

Now the velocity potential of an ellipsoid moving parallel to an axis is known. This velocity potential, with a suitable relation between k and $\dot{Z}$, will make the normal velocity at the surface of the ellipsoid continuous with the normal velocity of the rotational motion inside the ellipsoid, but it does not make the pressure continuous. Hence, if fluid can move outside the ellipsoid continuously with the rotational motion inside (described in section 1 above), then the motion outside the ellipsoid must be a rotational motion.

5. It cannot be argued that the application of HELMHOLTZ's method to determine the whole motion from the distribution of vortices inside the ellipsoid must determine an irrotational motion outside the ellipsoid continuous with the rotational motion inside, because HELMHOLTZ's method determines the irrotational motion by means of the distribution of vortices only when that distribution is known throughout space. This is not the case in the problem under discussion. For here the rotationally moving liquid has been arbitrarily limited by rejecting all the vortices outside the ellipsoid, and it is not known beforehand that the rejection of these vortices is possible.

6. Yet, on account of the interest of the problem, the paper contains a calculation of the velocity components in HELMHOLTZ's manner, supposing the only vortices to be those inside the ellipsoid, *i.e.*, starting from the values of the velocity components

$$u = \frac{2k}{c^2} x (z - Z),$$

$$v = \frac{2k}{c^2} y (z - Z),$$

$$w = \dot{Z} - \frac{2k}{a^2} (2x^2 + 2y^2 - a^2) - 2 \frac{k}{c^2} (z - Z)^2,$$

the components of the molecular rotation are first found, viz.:—

$$\xi = - k \left(\frac{4}{a^2} + \frac{1}{c^2}\right) y,$$

$$\eta = k \left(\frac{4}{a^2} + \frac{1}{c^2}\right) x,$$

$$\zeta = 0.$$

Then the potentials L, M, N of distributions of matter of densities $\frac{\xi}{2\pi}, \frac{\eta}{2\pi}, \frac{\zeta}{2\pi}$ respectively throughout the ellipsoid are determined.

These are, outside the ellipsoid,

$$L = -\tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

$$M = \tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)x\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

$$N = 0,$$

where ϵ is the parameter of the confocal ellipsoid through x, y, z.

Then

$$\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)x(z-Z)\int_\epsilon^\infty\frac{du}{(a^2+u)^2(c^2+u)^{3/2}},$$

$$\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)y(z-Z)\int_\epsilon^\infty\frac{du}{(a^2+u)^2(c^2+u)^{3/2}},$$

$$\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y} = ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_\epsilon^\infty\left(1-\frac{2r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}}.$$

To obtain the corresponding expressions inside the ellipsoid, it is necessary to replace ϵ by zero.

Outside the ellipsoid $\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}$, $\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}$ are the differential coefficients of the potential function

$$-\tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)(z-Z)\int_\epsilon^\infty\left(1-\frac{r^2}{a^2+u}-\frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)(c^2+u)^{3/2}},$$

which, with a suitable value of k, gives the potential of the irrotational motion outside the ellipsoid moving parallel to the axis z with velocity $\dot{Z}$.

But inside the ellipsoid $\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z}$, $\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}$, $\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y}$ are not respectively equal to the values of u, v, w, from which the investigation commenced.

In fact

$$u = \frac{\partial P}{\partial x}+\frac{\partial N}{\partial y}-\frac{\partial M}{\partial z},$$

$$v = \frac{\partial P}{\partial y}+\frac{\partial L}{\partial z}-\frac{\partial N}{\partial x}$$

$$w = \frac{\partial P}{\partial z}+\frac{\partial M}{\partial x}-\frac{\partial L}{\partial y},$$

where P is the potential function

$$\left[\frac{k}{c^2} - \tfrac{1}{2}ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}\right]\{r^2(z-\mathrm{Z}) - \tfrac{2}{3}(z-\mathrm{Z})^3\}$$

$$+\left[\dot{\mathrm{Z}} + 2k - ka^4c\left(\frac{4}{a^2}+\frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}}\right](z-\mathrm{Z}).$$

7. The expressions $\frac{\partial \mathrm{N}}{\partial y} - \frac{\partial \mathrm{M}}{\partial z}$, $\frac{\partial \mathrm{L}}{\partial z} - \frac{\partial \mathrm{N}}{\partial x}$, $\frac{\partial \mathrm{M}}{\partial x} - \frac{\partial \mathrm{L}}{\partial y}$ cannot be taken by themselves to represent the velocities inside and outside the ellipsoid, for, though they would furnish continuous values of the velocities at the surface of the ellipsoid, they would not make the pressure continuous.

Art. 1. *The Equations of Motion.*

If the velocity components of a mass of incompressible fluid at the point x, y, z be u, v, w at time t; if the pressure be p, the density ρ, and the potential of the impressed forces V, then the equations of motion are

$$\left.\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} &= -\frac{\partial}{\partial x}\left(\frac{p}{\rho}+\mathrm{V}\right)\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} &= -\frac{\partial}{\partial y}\left(\frac{p}{\rho}+\mathrm{V}\right)\\ \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho}+\mathrm{V}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{I.}),$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{II.}).$$

If the motion be symmetrical with regard to the axis of z, let $r = (x^2 + y^2)^{1/2}$, and let the velocity perpendicular to the axis and away from it be τ.

Then

$$\left.\begin{aligned}u &= \tau x/r\\ v &= \tau y/r\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{III.}),$$

and the equations of motion become

$$\left.\begin{aligned}\frac{\partial \tau}{\partial t} + \tau\frac{\partial \tau}{\partial r} + w\frac{\partial \tau}{\partial z} &= -\frac{\partial}{\partial r}\left(\frac{p}{\rho}+\mathrm{V}\right)\\ \frac{\partial w}{\partial t} + \tau\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho}+\mathrm{V}\right)\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{IV.}),$$

$$\frac{\partial \tau}{\partial r} + \frac{\tau}{r} + \frac{\partial w}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{V.}).$$

These are equivalent, on elimination of $\frac{p}{\rho} + \mathrm{V}$, to

$$\left(\frac{\partial}{\partial t} + \tau\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}\right)\left[\frac{1}{r}\left(\frac{\partial \tau}{\partial z} - \frac{\partial w}{\partial r}\right)\right] = 0 \quad \text{(VI.)},$$

and

$$\frac{\partial}{\partial r}(r\tau) + \frac{\partial}{\partial z}(rw) = 0 \quad \text{(VII.)}.$$

Art. 2. *The Equation satisfied by the Current Function.*

From equation (VII.) it follows that a function ψ exists, such that

$$\left.\begin{aligned} r\tau &= \partial\psi/\partial z \\ rw &= -\,\partial\psi/\partial r \end{aligned}\right\} \quad \text{(VIII.)}.$$

Substituting in (VI.), it follows that

$$\left(\frac{\partial}{\partial t} + \frac{1}{r}\frac{\partial\psi}{\partial z}\frac{\partial}{\partial r} - \frac{1}{r}\frac{\partial\psi}{\partial r}\frac{\partial}{\partial z}\right)\left[\frac{1}{r^2}\left(\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r}\right)\right] = 0 \quad \text{(IX.)}.$$

Hence, the whole motion depends on the current function ψ defined by (IX.).

Art. 3. *The Particular Integral selected.*

The following is a particular integral of (IX.) :—

$$\frac{\partial^2\psi}{\partial z^2} + \frac{\partial^2\psi}{\partial r^2} - \frac{1}{r}\frac{\partial\psi}{\partial r} = \left(\frac{8k}{a^2} + \frac{2k}{c^2}\right)r^2 \quad \text{(X.)},$$

where a, c, k are constants.

A particular integral of (X) is

$$\psi = r^2\left\{\frac{k}{a^2}(r^2 - a^2) + \frac{k}{c^2}(z - \mathrm{Z})^2 + f(t)\right\} \quad \text{(XI.)},$$

where Z and $f(t)$ are functions of t only.

Substituting this value of ψ in (VIII.),

$$\tau = 2\frac{k}{c^2}r(z - \mathrm{Z}),$$

$$w = -\,2\frac{k}{c^2}(z - \mathrm{Z})^2 - 2\frac{k}{a^2}(2r^2 - a^2) - 2f(t).$$

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Art. 4. *The Surfaces which contain the same particles of fluid.*

The next step is to find the surfaces which contain the same particles of fluid throughout the motion.

If $\lambda =$ const. be one family of these surfaces,

$$\frac{\partial \lambda}{\partial t} + \tau \frac{\partial \lambda}{\partial r} + w \frac{\partial \lambda}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XII.}).$$

Therefore

$$\frac{\partial \lambda}{\partial t} + 2\frac{k}{c^2} r (z - Z) \frac{\partial \lambda}{\partial r} - \left[2\frac{k}{c^2}(z - Z)^2 + 2\frac{k}{a^2}(2r^2 - a^2) + 2f(t) \right] \frac{\partial \lambda}{\partial z} = 0 \quad (\text{XIII.}).$$

The auxiliary equations for λ are

$$\frac{dt}{1} = \frac{dr}{2\frac{k}{c^2} r (z - Z)} = \frac{dz}{-2\frac{k}{c^2}(z - Z)^2 - 2\frac{k}{a^2}(2r^2 - a^2) - 2f(t)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XIV.}),$$

$$= \frac{-2\frac{k}{c^2} r^2 (z - Z) \frac{dZ}{dt} dt + \left[2\frac{k}{c^2} r (z - Z)^2 + 2\frac{k}{a^2} r (2r^2 - a^2) \right] dr + 2\frac{k}{c^2} r^2 (z - Z)\, dz}{-2\frac{k}{c^2} r^2 (z - Z) \left[\frac{dZ}{dt} + 2f(t) \right]}$$

$$= \frac{d\left[r^2 \left\{ \frac{k}{c^2}(z - Z)^2 + \frac{k}{a^2}(r^2 - a^2) \right\} \right]}{-2\frac{k}{c^2} r^2 (z - Z) \left[\frac{dZ}{dt} + 2f(t) \right]}.$$

Hence if $f(t) = -\frac{1}{2}\frac{dZ}{dt} = -\frac{1}{2}\dot{Z}$, one solution of (XIII.) will be

$$\lambda = kr^2 \left[\frac{r^2}{a^2} + \frac{(z - Z)^2}{c^2} - 1 \right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XV.}).$$

Hence the component velocities

$$\left. \begin{aligned} \tau &= 2\frac{k}{c^2} r (z - Z) \\ w &= \dot{Z} - 2\frac{k}{a^2}(2r^2 - a^2) - 2\frac{k}{c^2}(z - Z)^2 \end{aligned} \right\} \quad . \quad . \quad . \quad . \quad (\text{XVI.}),$$

belong to a motion in which the surfaces $\lambda =$ const. given by (XV.) contain the same particles of fluid throughout the motion.

Also by (XI.)

$$\psi = r^2\left[\frac{k}{a^2}(r^2 - a^2) + \frac{k}{c^2}(z - Z)^2 - \tfrac{1}{2}\dot{Z}\right] \quad . \quad . \quad . \quad . \quad \text{(XVII.)}.$$

Art. 5. *The Pressure.*

Substituting the above values of τ and w in equations (IV.), they become

$$\left.\begin{aligned} -\frac{4k^2}{a^2c^2}r(2r^2 - a^2) &= -\frac{\partial}{\partial r}\left(\frac{p}{\rho} + V\right) \\ \ddot{Z} + \frac{8k^2}{c^4}(z - Z)^3 - \frac{8k^2}{c^2}(z - Z) &= -\frac{\partial}{\partial z}\left(\frac{p}{\rho} + V\right) \end{aligned}\right\} \quad . \quad . \quad \text{(XVIII.)}.$$

Therefore

$$\frac{p}{\rho} + V = \frac{2k^2}{a^2c^2}\left(r^2 - \frac{a^2}{2}\right)^2 - (z - Z)\ddot{Z} - \frac{2k^2}{c^4}(z - Z)^4 + \frac{4k^2}{c^2}(z - Z)^2$$
$$+ \text{ an arbitrary function of } t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XIX.)}.$$

Art. 6. *The Molecular Rotation.*

If 2ω be the molecular rotation,

$$2\omega = \frac{\partial \tau}{\partial z} - \frac{\partial w}{\partial r} = \left(\frac{8k}{a^2} + \frac{2k}{c^2}\right)r.$$

Therefore

$$\omega = \left(\frac{4k}{a^2} + \frac{k}{c^2}\right)r. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XX.)}.$$

Hence the molecular rotation varies as the distance from the axis of symmetry.

The vortex lines are circles, whose centres are on the axis of symmetry, and whose planes are perpendicular to it.

Art. 7. *Further simplification of the Particular Integral*

Amongst the surfaces given by making λ constant in XV., there is one, viz. :—

$$kr^2\left[\frac{r^2}{a^2} + \frac{(z - Z)^2}{c^2} - 1\right] = 0,$$

which breaks up into the evanescent cylinder

$$r^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(XXI.)},$$

and the ellipsoid of revolution,

$$\frac{r^2}{a^2} + \frac{(z-Z)^2}{c^2} - 1 = 0.$$

If, further, it be supposed that $c = a$, the ellipsoid becomes the sphere

$$r^2 + (z - Z)^2 = a^2 \quad \text{(XXII.)}.$$

The discussion will now be limited to this case.

In it

$$\left.\begin{aligned} \tau &= 2\frac{k}{a^2} r (z - Z) \\ w &= \dot{Z} - 2\frac{k}{a^2}(2r^2 - a^2) - 2\frac{k}{a^2}(z-Z)^2 \end{aligned}\right\} \quad \text{(XXIII.)}$$

$$\omega = \frac{5kr}{a^2} \quad \text{(XXIV.)},$$

$$\frac{p}{\rho} + V = \frac{2k^2}{a^4}\left(r^2 - \frac{a^2}{2}\right)^2 - (z-Z)\ddot{Z} - \frac{2k^2}{a^4}(z-Z)^4 + \frac{4k^2}{a^2}(z-Z)^2 + \frac{\Pi}{\rho}. \quad \text{(XXV.)},$$

where Π/ρ is an arbitrary function of t.

$$\left.\begin{aligned} \psi &= r^2\left[\frac{k}{a^2}\{r^2 + (z-Z)^2 - a^2\} - \tfrac{1}{2}\dot{Z}\right] \\ {}^*\lambda &= \frac{k}{a^2} r^2 \{r^2 + (z-Z)^2 - a^2\} \end{aligned}\right\} \quad \text{(XXVI.)}.$$

* The surfaces $\lambda =$ const. are a particular case of some surfaces that were noticed by Professor LAMB in a paper "On the Vibrations of an Elastic Sphere," published in the 'Proceedings of the London Mathematical Society,' vol. 13, p. 205.

In equation 75 of that paper, viz.,

$$\psi = \tfrac{1}{2}\varpi^2\{\psi_1(kr) - \psi_1(ka)\},$$

where

$$\psi_1(z) = 1 - \frac{z^2}{2.5} + \frac{z^4}{2.4.5.7} - \ldots,$$

the current function may be written

$$C\varpi^2\{\psi_1(kr) - \psi_1(ka)\} = C\varpi^2\left[-\frac{k^2}{2.5}(r^2 - a^2) + \frac{k^4}{2.4.5.7}(r^4 - a^4) - \ldots\right].$$

If we suppose Ck^2 to be finite, but $k = 0$, this becomes

$$C'\varpi^2(r^2 - a^2),$$

or, in the notation of this paper,

$$C'r^2\{r^2 + (z-Z)^2 - a^2\},$$

which agrees with the above.

Hence, at the surface of the sphere (XXII.), putting

$$\left.\begin{aligned} r &= a \sin \theta \\ z - \text{Z} &= a \cos \theta \end{aligned}\right\} \quad \text{(XXVII.)},$$

$$\tau = 2k \sin \theta \cos \theta \quad \text{(XXVIII.)},$$

$$w = \dot{\text{Z}} - 2k \sin^2 \theta \quad \text{(XXIX.)},$$

$$\frac{p}{\rho} + \text{V} = 2k^2 \cos^2 \theta + \tfrac{1}{2} k^2 - a \cos \theta \ddot{\text{Z}} + \frac{\Pi}{\rho} \quad \text{(XXX.)},$$

Art. 8. *The Irrotational Motion outside the Sphere.*

The velocity potential of a sphere of radius a, moving with velocity $\dot{\text{Z}}$ parallel to the axis of z, at external points, is

$$\phi = -a^3 \dot{\text{Z}} (z - \text{Z})/(2\text{R}^3) = -a^3 \dot{\text{Z}} \cos \theta/(2\text{R}^2) \quad \text{(XXXI.)},$$

where

$$\text{R}^2 = r^2 + (z - \text{Z})^2$$

(see BASSET'S 'Hydrodynamics,' vol. I., Art. 143).

Whence

$$\frac{\partial \phi}{\partial r} = 3a^3 \dot{\text{Z}} r (z - \text{Z})/(2\text{R}^5) \quad \text{(XXXII.)},$$

$$\frac{\partial \phi}{\partial z} = a^3 \dot{\text{Z}} \{3 (z - \text{Z})^2 - \text{R}^2\}/(2\text{R}^5) \quad \text{(XXXIII.)},$$

$$\begin{aligned} \frac{p}{\rho} + \text{V} = {} & a^3 [\text{R}^2 \{(z - \text{Z}) \ddot{\text{Z}} - \dot{\text{Z}}^2\} + 3 (z - \text{Z})^2 \dot{\text{Z}}^2]/(2\text{R}^5) \\ & - a^6 \dot{\text{Z}}^2 [\text{R}^2 + 3 (z - \text{Z})^2]/(8\text{R}^8) + \text{T} \end{aligned} \quad \text{(XXXIV.)},$$

where T is an arbitrary function of t.

Hence, at a point on the surface of the sphere (XXII.),

$$\frac{\partial \phi}{\partial r} = \tfrac{3}{2} \dot{\text{Z}} \sin \theta \cos \theta. \quad \text{(XXXV.)},$$

$$\frac{\partial \phi}{\partial z} = \dot{\text{Z}} (1 - \tfrac{3}{2} \sin^2 \theta) \quad \text{(XXXVI.)},$$

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{2} a \cos\theta \, \ddot{\mathrm{Z}} - \tfrac{5}{8} \dot{\mathrm{Z}}^2 + \tfrac{9}{8} \cos^2\theta \, \dot{\mathrm{Z}}^2 + \mathrm{T} \quad . \quad . \quad (\text{XXXVII.}).$$

The value of the current function ψ, corresponding to the velocity potential ϕ of (XXXI.) is

$$\psi = - a^3 \dot{\mathrm{Z}} r^2 / (2\mathrm{R}^3) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XXXVIII.}).$$

If $\lambda =$ const. be a family of surfaces containing the same particles of fluid

$$\frac{\partial\lambda}{\partial t} + \frac{1}{r} \frac{\partial\psi}{\partial z} \frac{\partial\lambda}{\partial r} - \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial\lambda}{\partial z} = 0 \quad . \quad . \quad . \quad . \quad . \quad (\text{XXXIX.}).$$

An integral of this equation is

$$\lambda = \psi + \frac{r^2}{2} \mathrm{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XL.}),$$

for $\dot{\mathrm{Z}}$ being constant.

$$\frac{\partial\lambda}{\partial t} = \frac{\partial\psi}{\partial t} = \frac{\partial\psi}{\partial z} (- \dot{\mathrm{Z}}),$$

$$\frac{\partial\lambda}{\partial r} = \frac{\partial\psi}{\partial r} + r\dot{\mathrm{Z}},$$

$$\frac{\partial\lambda}{\partial z} = \frac{\partial\psi}{\partial z},$$

therefore

$$\frac{\partial\lambda}{\partial t} + \frac{1}{r} \frac{\partial\psi}{\partial z} \frac{\partial\lambda}{\partial r} - \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial\lambda}{\partial z}$$
$$= - \dot{\mathrm{Z}} \frac{\partial\psi}{\partial z} + \frac{1}{r} \frac{\partial\psi}{\partial z} \left(\frac{\partial\psi}{\partial r} + r\dot{\mathrm{Z}} \right) - \frac{1}{r} \frac{\partial\psi}{\partial r} \frac{\partial\psi}{\partial z}$$
$$= 0.$$

Hence the surfaces $\lambda =$ const. are

$$\frac{r^2}{2} \dot{\mathrm{Z}} \left(1 - \frac{a^3}{\mathrm{R}^3} \right) = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XLI.}).$$

Art. 9. *The Conditions for the continuity of the rotational and irrotational motions.*

In order that the motion inside the sphere (XXII.) may be continuous with that outside, the equations (XXVIII.) and (XXXV.) must make $\tau = \partial\phi/\partial r$.

Therefore

$$2k = \tfrac{3}{2} \dot{\mathrm{Z}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XLII.}).$$

The equations (XXIX.) and (XXXVI.) must make $w = \partial\phi/\partial z$.

This leads again to (XLII.).

The equations (XXX.) and (XXXVII.) must give the same value for $p/\rho + \mathrm{V}$. This requires that

$$\ddot{\mathrm{Z}} = 0,$$

$$2k^2 = \tfrac{9}{8}\dot{\mathrm{Z}}^2,$$

and

$$\mathrm{T} = \tfrac{5}{8}\dot{\mathrm{Z}}^2 + \tfrac{1}{2}k^2 + \frac{\Pi}{\rho}.$$

The first and second of these follow from (XLII.).

The last gives

$$\mathrm{T} = \tfrac{29}{32}\dot{\mathrm{Z}}^2 + \frac{\Pi}{\rho}.$$

Hence (XXXIV.) can be written

$$\frac{p}{\rho} + \mathrm{V} = a^3\dot{\mathrm{Z}}^2[3(z-\mathrm{Z})^2 - \mathrm{R}^2]/(2\mathrm{R}^5) - a^6\dot{\mathrm{Z}}^2[3(z-\mathrm{Z})^2 + \mathrm{R}^2]/(8\mathrm{R}^8) + \tfrac{29}{32}\dot{\mathrm{Z}}^2 + \frac{\Pi}{\rho}.$$

Therefore

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{8}\dot{\mathrm{Z}}^2\left[\left\{5 - 4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6\right\} + 3\cos^2\theta\left\{4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6\right\} + \tfrac{9}{4}\right] + \frac{\Pi}{\rho} \quad \text{(XLIII.)}.$$

Hence at the surface of the sphere

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{8}\dot{\mathrm{Z}}^2(9\cos^2\theta + \tfrac{9}{4}) + \frac{\Pi}{\rho} \quad . \quad . \quad . \quad . \quad . \quad \text{(XLIV.)}.$$

Further, outside the sphere $\mathrm{R} > a$, therefore,

$$5 - 4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6 > 0$$

$$4\left(\frac{a}{\mathrm{R}}\right)^3 - \left(\frac{a}{\mathrm{R}}\right)^6 > 0,$$

therefore,

$$\frac{p}{\rho} + \mathrm{V} > \frac{\Pi}{\rho}.$$

Now using the value $k = \frac{3}{4}\dot{\mathrm{Z}}$ from (XLII.), putting $\ddot{\mathrm{Z}} = 0$, equations (XXIII.) and (XXV.) give inside the sphere

$$\left.\begin{aligned} \tau &= 3\dot{Z}r(z-Z)/(2a^2) \\ w &= \dot{Z}\{5a^2 - 3(z-Z)^2 - 6r^2\}/(2a^2) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\text{XLV.})$$

$$\frac{p}{\rho} + V = \frac{9\dot{Z}^2}{8a^4}\left[\left(r^2 - \frac{a^2}{2}\right)^2 - \{(z-Z)^2 - a^2\}^2 + a^4\right] + \frac{\Pi}{\rho} \quad . \quad (\text{XLVI.}).$$

Also from (XXVI.)

$$\psi = 3\dot{Z}r^2[R^2 - \tfrac{5}{3}a^2]/(4a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XLVII.}).$$

and

$$\lambda = 3\dot{Z}r^2[R^2 - a^2]/(4a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XLVIII.}).$$

Also from (XXIV.)

$$\omega = 15\dot{Z}r/(4a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{XLIX}).$$

It may be noted that the value of $p/\rho + V$ given by (XLVI.) is least when $(r^2 - \frac{1}{2}a^2)^2$ is least, and $\{(z-Z)^2 - a^2\}^2$ is greatest, *i.e.*, when $r^2 = \frac{1}{2}a^2$, and $z - Z = 0$; and then $p/\rho + V = \Pi/\rho$.

Hence Π/ρ is the minimum value of $p/\rho + V$ throughout the whole mass of moving fluid.

Further, all points on the circle $r = a/\sqrt{2}$, $z = Z$ represent the surface

$$\lambda = -3\dot{Z}a^2/(16);$$

for this surface is

$$r^2(R^2 - a^2) = -a^4/4, \quad \textit{i.e.}, \quad (r^2 - \tfrac{1}{2}a^2)^2 + r^2(z-Z)^2 = 0.$$

A neighbouring surface is

$$(r^2 - \tfrac{1}{2}a^2)^2 + r^2(z-Z)^2 = 2\epsilon^4,$$

where ϵ is small.

Putting

$$r = r' + a \,.\, 2^{-\frac{1}{2}}$$
$$z = z' + Z$$

and retaining only the principal terms, it becomes

$$\frac{r'^2}{(\epsilon^2/a)^2} + \frac{z'^2}{(2\epsilon^2/a)^2} = 1,$$

proving that the section by a plane through the axis of z is an infinitely small ellipse, with its major axis double the minor axis, the minor axis being perpendicular to the direction in which the vortex sphere moves.

Art. 10. *The Cyclic Constant of the Spherical Vortex.*

The cyclic constant of the vortex is

$$\int_{-a}^{+a}\int_{0}^{\sqrt{(a^2-z^2)}} 2\omega \, dz \, dr = \int_{-a}^{+a}\int_{0}^{\sqrt{(a^2-z^2)}} \frac{15\dot{Z}r}{2a^2} \, dz \, dr$$

$$= \frac{15\dot{Z}}{4a^2}\int_{-a}^{+a} (a^2 - z^2) \, dz$$

$$= \frac{15Z}{4a^2}\left[a^2 z - \frac{z^3}{3}\right]_{-a}^{+a}$$

$$= 5a\dot{Z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (L.)$$

Hence the cyclic constant of the vortex sphere is equal to five times the radius of the sphere multiplied by the uniform velocity with which the vortex sphere moves parallel to its axis.

Art. 11. *The Kinetic Energy of the Vortex.*

The kinetic energy of the vortex

$$= \pi\rho\int_{Z-a}^{Z+a} dz \int_{0}^{\sqrt{\{a^2-(z-Z)^2\}}} dr \, r \, (\tau^2 + w^2)$$

$$= \pi\rho\int_{Z-a}^{Z+a} dz \int_{0}^{\sqrt{\{a^2-(z-Z)^2\}}} dr \, . \, r \, . \, \frac{\dot{Z}^2}{4a^4}\left[\begin{array}{l}25a^4 - 30a^2(z-Z)^2 + 9(z-Z)^4 \\ \qquad + 45r^2(z-Z)^2 - 60a^2r^2 + 36r^4\end{array}\right]$$

$$= \frac{\pi\rho\dot{Z}^2}{8a^4}\int_{Z-a}^{Z+a} dz \left[\begin{array}{l}\{25a^4 - 30a^2(z-Z)^2 + 9(z-Z)^4\}\{a^2 - (z-Z)^2\} \\ + \tfrac{1}{2}\{45(z-Z)^2 - 60a^2\}\{a^2-(z-Z)^2\}^2 + 12\{a^2-(z-Z)^2\}^3\end{array}\right]$$

$$= \frac{\pi\rho\dot{Z}^2}{16a^4}\int_{Z-a}^{Z+a} dz \, \{14a^6 - 17a^4(z-Z)^2 + 3(z-Z)^6\}$$

$$= \frac{\pi\rho\dot{Z}^2}{8a^4}\{14a^7 - \tfrac{17}{3}a^7 + \tfrac{3}{7}a^7\}$$

$$= \frac{23\pi\rho\dot{Z}^2a^3}{21}.$$

The kinetic energy of the irrotational motion outside the vortex is

$$\pi\rho \int_a^\infty d\mathrm{R} \int_0^\pi d\theta \mathrm{R}^2 \sin\theta (\tau^2 + w^2)$$

$$= \pi\rho \int_a^\infty d\mathrm{R} \int_0^\pi d\theta \mathrm{R}^2 \sin\theta \frac{a^6\dot{\mathrm{Z}}^2}{4\mathrm{R}^6} (3\cos^2\theta + 1)$$

$$= \pi\rho \frac{a^6\dot{\mathrm{Z}}^2}{4} \cdot \frac{4}{3a^3} = \frac{\pi\rho a^3\dot{\mathrm{Z}}^2}{3}.$$

Art. 12. *The Distribution of Matter which would produce the Velocity Potential of the Irrotational Motion.*

The velocity potential $-a^3\dot{\mathrm{Z}}(z-\mathrm{Z})/(2\mathrm{R}^3)$ at points outside the sphere is due to a distribution of matter inside the sphere of density

$$-15\dot{\mathrm{Z}}(z-\mathrm{Z})/(8\pi a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{LI.}),$$

and the potential of this distribution of matter inside the sphere is

$$\dot{\mathrm{Z}}(z-\mathrm{Z})\{3\mathrm{R}^2 - 5a^2\}/(4a^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{LII.}).$$

For

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\dot{\mathrm{Z}}(z-\mathrm{Z})(3\mathrm{R}^2 - 5a^2)}{4a^2}\right) + 4\pi\left(-\frac{15\dot{\mathrm{Z}}(z-\mathrm{Z})}{8\pi a^2}\right) = 0. \quad (\mathrm{LIII.}).$$

Further, when $\mathrm{R} = a$

and

$$\left.\begin{aligned} \dot{\mathrm{Z}}(z-\mathrm{Z})\{3\mathrm{R}^2 - 5a^2\}/(4a^2) &= -\tfrac{1}{2}\dot{\mathrm{Z}}(z-\mathrm{Z}) \\ -a^3\dot{\mathrm{Z}}(z-\mathrm{Z})/(2\mathrm{R}^3) &= -\tfrac{1}{2}\dot{\mathrm{Z}}(z-\mathrm{Z}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{LIV.}).$$

Again, when $\mathrm{R} = a$

and

$$\left.\begin{aligned} \frac{\partial}{\partial r}\left[\frac{\dot{\mathrm{Z}}}{4a^2}(z-\mathrm{Z})(3\mathrm{R}^2 - 5a^2)\right] &= \frac{3\dot{\mathrm{Z}}}{2a^2} r(z-\mathrm{Z}) \\ \frac{\partial}{\partial r}[-a^3\dot{\mathrm{Z}}(z-\mathrm{Z})/(2\mathrm{R}^3)] &= \frac{3\dot{\mathrm{Z}}}{2a^2} r(z-\mathrm{Z}) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad (\mathrm{LV.})$$

Also when $\mathrm{R} = a$

and

$$\left.\begin{aligned} \frac{\partial}{\partial z}\left[\frac{\dot{\mathrm{Z}}}{4a^2}(z-\mathrm{Z})(3\mathrm{R}^2 - 5a^2)\right] &= -\frac{\dot{\mathrm{Z}}}{2} + \frac{3\dot{\mathrm{Z}}}{2a^2}(z-\mathrm{Z})^2 \\ \frac{\partial}{\partial z}[-a^3\dot{\mathrm{Z}}(z-\mathrm{Z})/(2\mathrm{R}^3)] &= -\frac{\dot{\mathrm{Z}}}{2} + \frac{3\dot{\mathrm{Z}}}{2a^2}(z-\mathrm{Z})^2 \end{aligned}\right\} \quad . \quad . \quad (\mathrm{LVI.}).$$

The equations (LIV.) show that the potential function in (LII.) is continuous with the velocity potential of (XXXI.) at the surface of the sphere. The equations (LV.) and (LVI.) show that the differential coefficients are also continuous. Finally (LIII.) shows that the density of the distribution of matter is that given in (LI.)

Art. 13. *Expression of the Velocity Components of the Rotational Motion in* CLEBSCH'S *Form.*

CLEBSCH has proved that the velocity components can be expressed as follows :—

$$\tau = \frac{\partial \chi}{\partial r} + \lambda \frac{\partial \mu}{\partial r} \quad \text{(LVII.)}$$

$$w = \frac{\partial \chi}{\partial z} + \lambda \frac{\partial \mu}{\partial z} \quad \text{(LVIII.)}$$

where

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \lambda = 0 \quad \text{(LIX.)}$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \mu = 0 \quad \text{(LX.)}$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \chi = -\left(\frac{p}{\rho} + \text{V}\right) + \tfrac{1}{2}(\tau^2 + w^2) \quad \text{(LXI.)}$$

The value of λ may be taken as

$$3\dot{\text{Z}}r^2 (\text{R}^2 - a^2)/(4a^2).$$

(See equation XLVIII.)

To find μ, there are the equations

$$\frac{dt}{1} = \frac{dr}{\tau} = \frac{dz}{w} = \frac{d\mu}{0} \quad \text{(LXII.)}$$

Therefore

$$\frac{dt}{1} = \frac{dr}{3\dot{\text{Z}}r(z - \text{Z})/(2a^2)} = \frac{dz}{\dot{\text{Z}}\{5a^2 - 3(z - \text{Z})^2 - 6r^2\}/(2a^2)} = \frac{d\mu}{0} \quad \text{(LXIII.)}$$

One integral of (LXIII.) is

$$\lambda = \text{constant},$$

i.e.,

$$3\dot{\text{Z}}r^2 \{\text{R}^2 - a^2\}/(4a^2) = 3\dot{\text{Z}}\text{L}/(4a^2) \quad \text{(LXIV.)},$$

where L is some constant.

From (LXIV.) it follows that

$$r(z - Z) = \sqrt{(L + r^2a^2 - r^4)} \quad \text{(LXV.)}$$

Substituting in (LXIII.)

$$\frac{3\dot{Z}}{2a^2}\,dt = \frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} \quad \text{(LXVI.)}$$

Therefore

$$\int \frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} - \frac{3Z}{2a^2} = \text{constant} \quad \text{(LXVII.)}.$$

Hence

$$\mu = C\left[\int \frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} - \frac{3Z}{2a^2}\right] \quad \text{(LXVIII.)},$$

where, after the integration is performed, L must be replaced by $r^2\{R^2 - a^2\}$.

To determine C, it is necessary to substitute in the equation

$$\frac{\partial\tau}{\partial z} - \frac{\partial w}{\partial r} = \frac{\partial\lambda}{\partial z}\frac{\partial\mu}{\partial r} - \frac{\partial\lambda}{\partial r}\frac{\partial\mu}{\partial z} \quad \text{(LXIX.)},$$

i.e.,

$$\frac{3\dot{Z}}{2a^2}r + \frac{\dot{Z}}{2a^2}(12r) = \frac{3\dot{Z}}{4a^2}2r^2(z - Z)\left[\frac{C}{r(z - Z)} - C\{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right]$$
$$- \frac{3\dot{Z}}{4a^2}\{2r(R^2 - a^2) + 2r^3\}\left[-Cr^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right].$$

Therefore

$$\frac{3\dot{Z}}{2a^2}(5r) = \frac{3\dot{Z}}{2a^2}rC.$$

Therefore

$$C = 5.$$

Hence

$$\mu = 5\int\frac{dr}{\sqrt{(L + r^2a^2 - r^4)}} - \frac{15Z}{2a^2} \quad \text{(LXX.)}.$$

Hence

$$\lambda\frac{\partial\mu}{\partial r} = \frac{15\dot{Z}}{4a^2}r^2(R^2 - a^2)\left[\frac{1}{r(z - Z)} - \{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right]. \quad \text{(LXXI.)}.$$

$$\lambda\frac{\partial\mu}{\partial z} = \frac{15\dot{Z}}{4a^2}r^2(R^2 - a^2)\left[-r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad \text{(LXXII.)}$$

Therefore

$$\frac{\partial\chi}{\partial r} = \tau - \lambda\frac{\partial\mu}{\partial r}$$

$$= \frac{3\dot{Z}}{2a^2}r(z-Z)$$

$$-\frac{15\dot{Z}}{4a^2}r^2(R^2-a^2)\left[\frac{1}{r(z-Z)} - \{r(R^2-a^2)+r^3\}\int\frac{dr}{(L+r^2a^2-r^4)^{3/2}}\right] \quad \text{(LXXIII.)},$$

$$\frac{\partial\chi}{\partial z} = w - \lambda\frac{\partial\mu}{\partial z}$$

$$= \frac{\dot{Z}}{2a^2}\{5a^2-3(z-Z)^2-6r^2\} + \frac{15\dot{Z}}{4a^2}r^4(R^2-a^2)(z-Z)\int\frac{dr}{(L+r^2a^2-r^4)^{3/2}} \quad \text{(LXXIV.)}$$

Next, $\partial\chi/\partial t$ can be found by means of (LXI.)

$$\frac{\partial\chi}{\partial t} = -\left(\frac{p}{\rho}+V\right) + \tfrac{1}{2}(\tau^2+w^2) - \tau\frac{\partial\chi}{\partial r} - w\frac{\partial\chi}{\partial z}$$

$$= -\left(\frac{p}{\rho}+V\right) + \tfrac{1}{2}(\tau^2+w^2) - \tau\left(\tau-\lambda\frac{\partial\mu}{\partial r}\right) - w\left(w-\lambda\frac{\partial\mu}{\partial z}\right)$$

$$= -\left(\frac{p}{\rho}+V\right) - \tfrac{1}{2}(\tau^2+w^2) + \lambda\left(\tau\frac{\partial\mu}{\partial r}+w\frac{\partial\mu}{\partial z}\right)$$

$$= -\left(\frac{p}{\rho}+V\right) - \tfrac{1}{2}(\tau^2+w^2) - \lambda\frac{\partial\mu}{\partial t}$$

$$= -\frac{\Pi}{\rho} - \frac{9\dot{Z}^2}{8a^4}\left[r^4-r^2a^2+\frac{a^4}{4}-(z-Z)^4+2a^2(z-Z)^2\right]$$

$$-\frac{9\dot{Z}^2}{8a^4}[r^2(z-Z)^2]$$

$$-\frac{\dot{Z}^2}{8a^4}[25a^4-30a^2(z-Z)^2-60r^2a^2+9(z-Z)^4+36(z-Z)^2r^2+36r^4]$$

$$+\frac{3\dot{Z}}{4a^2}r^2(R^2-a^2)\left[\frac{15\dot{Z}}{2a^2} - 5\dot{Z}r^2(z-Z)\int\frac{dr}{(L+r^2a^2-r^4)^{3/2}}\right],$$

therefore

$$\frac{\partial\chi}{\partial t} = -\frac{\Pi}{\rho} - \frac{\dot{Z}^2}{8a^2}\left[\frac{109}{4}a^2-24r^2-12(z-Z)^2\right]$$

$$-\frac{15\dot{Z}^2r^4(R^2-a^2)(z-Z)}{4a^2}\int\frac{dr}{(L+r^2a^2-r^4)^{3/2}} \quad \text{. . . . . . (LXXV.)}$$

Next taking U as the potential of the distribution of matter inside the sphere which would produce the potential of the irrotational motion outside the sphere

$$U = \dot{Z}(z - Z)\{3R^2 - 5a^2\}/(4a^2)$$

by equation (LII.), therefore

$$\frac{\partial U}{\partial r} = \frac{3\dot{Z}}{4a^2}(z - Z)\,2r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{LXXVI.}),$$

$$\frac{\partial U}{\partial z} = \frac{3\dot{Z}}{4a^2}\left[2(z - Z)^2 + R^2\right] - \frac{5\dot{Z}}{4} \quad . \quad . \quad . \quad . \quad . \quad (\text{LXXVII.}),$$

$$\frac{\partial U}{\partial t} = -\frac{3\dot{Z}^2}{4a^2}\left[2(z - Z)^2 + R^2\right] + \frac{5\dot{Z}^2}{4} \quad . \quad . \quad . \quad (\text{LXXVIII.}).$$

Hence

$$\frac{\partial(\chi - U)}{\partial r} = -5\lambda\left[\frac{1}{r(z - Z)} - \{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] . \quad (\text{LXXIX.}),$$

$$\frac{\partial(\chi - U)}{\partial z} = -5\lambda\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad . \quad . \quad . \quad . \quad . \quad (\text{LXXX.}),$$

$$\frac{\partial(\chi - U)}{\partial t} = -\frac{\Pi}{\rho} - \tfrac{29}{32}\dot{Z}^2 + 5\lambda\dot{Z}\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad (\text{LXXXI.}),$$

therefore

$$\frac{\partial}{\partial r}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$
$$= -5\lambda\left[\frac{1}{r(z - Z)} - \{r(R^2 - a^2) + r^3\}\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad (\text{LXXXII.}),$$

$$\frac{\partial}{\partial z}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$
$$= -5\lambda\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad . \quad . \quad (\text{LXXXIII.}),$$

$$\frac{\partial}{\partial t}\left[\chi - U + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{Z}^2\right)dt\right]$$
$$= 5\lambda\dot{Z}\left[\frac{1}{r^2} - r^2(z - Z)\int\frac{dr}{(L + r^2a^2 - r^4)^{3/2}}\right] \quad . \quad . \quad . \quad (\text{LXXXIV.}).$$

From (LXXXIII.) and (LXXXIV.) it follows that

$$\frac{\partial}{\partial t}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right] = -\dot{\mathrm{Z}}\frac{\partial}{\partial z}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right].$$

Hence $\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt$ is a function of r and $z - \mathrm{Z}$ only, therefore

$$\frac{\partial}{\partial(z-\mathrm{Z})}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right]$$
$$= -5\lambda\left[\frac{1}{r^2} - r^2(z-\mathrm{Z})\int\frac{dr}{(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right] \quad . \quad . \quad \text{(LXXXV.)}.$$

Before proceeding further it is necessary to prove that

$$\int\frac{dr}{(\mathrm{L} + r^2a^2 - r^4)^{3/2}}$$
$$= \frac{1}{r^3(\mathrm{L} + r^2a^2 - r^4)^{1/2}} + 4\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \int\frac{\mathrm{L}\,dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}} \quad . \quad \text{(LXXXVI.)}.$$

Differentiating both sides with regard to r, an identity is obtained.

Hence the result holds.

Making use of (LXXXVI.) in (LXXXII.), and remembering that after the integrations in (LXXXVI.) are effected, L may be replaced by $r^2(\mathrm{R}^2 - a^2)$,

$$\frac{\partial}{\partial r}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r(z-\mathrm{Z})} - \frac{r(\mathrm{R}^2 - a^2) + r^3}{r^4(z-\mathrm{Z})} - \frac{1}{2}\frac{\partial \mathrm{L}}{\partial r}\left\{4\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \int\frac{\mathrm{L}\,dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right\}\right]$$

$$= -5\lambda\left[-\frac{\mathrm{L}}{r^4\sqrt{(\mathrm{L} + r^2a^2 - r^4)}} - \frac{\partial \mathrm{L}}{\partial r}\left\{2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \mathrm{L}\int\frac{dr}{2r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right\}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\left[\frac{\mathrm{L}^2}{r^4\sqrt{(\mathrm{L} + r^2a^2 - r^4)}} - \mathrm{L}^2\frac{\partial \mathrm{L}}{\partial r}\int\frac{dr}{2r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}} + 2\mathrm{L}\frac{\partial \mathrm{L}}{\partial r}\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\frac{\partial}{\partial r}\left[\mathrm{L}^2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(LXXXVII.)}.$$

Also

$$\frac{\partial}{\partial (z - \mathrm{Z})}\left[\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt\right]$$

$$= -5\lambda\left[\frac{1}{r^2} - r^2(z - \mathrm{Z})\int\frac{dr}{(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right]$$

$$= -5\lambda\left[\frac{1}{r^2} - r^2(z - \mathrm{Z})\left\{\frac{1}{r^3(\mathrm{L} + r^2a^2 - r^4)^{1/2}} + 4\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \int\frac{\mathrm{L}\,dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right\}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\,\mathrm{L}\,\frac{\partial \mathrm{L}}{\partial z}\left\{2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} - \mathrm{L}\int\frac{1}{2}\,\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{3/2}}\right\}$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\left[\frac{\partial \mathrm{L}^2}{\partial z}\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} + \mathrm{L}^2\frac{\partial}{\partial z}\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\frac{\partial}{\partial z}\left[\mathrm{L}^2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right]$$

$$= \frac{15\dot{\mathrm{Z}}}{4a^2}\frac{\partial}{\partial (z - \mathrm{Z})}\left[\mathrm{L}^2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}}\right] \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(LXXXVIII.)}.$$

Now by (LXXXVII.) and (LXXXVIII.)

$$\chi - \mathrm{U} + \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt = \frac{15\dot{\mathrm{Z}}}{4a^2}\,\mathrm{L}^2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} + \text{const.}$$

Therefore

$$\chi = \mathrm{U} - \int\left(\frac{\Pi}{\rho} + \tfrac{29}{32}\dot{\mathrm{Z}}^2\right)dt + \frac{20a^2}{3\dot{\mathrm{Z}}}\lambda^2\int\frac{dr}{r^4(\mathrm{L} + r^2a^2 - r^4)^{1/2}} + \text{const.} \quad . \quad \text{(LXXXIX.)},$$

where, after the integration has been performed, L must be replaced by

$$4a^2\lambda/(3\dot{\mathrm{Z}}).$$

Art. 14. *The Figure.*

The figure has been constructed from the two following tables.

Table I. gives the form of the surfaces

$$^{*}r^2(\mathrm{R}^2 - a^2) = -d^4,$$

which are inside the sphere, and which always contain the same particles of fluid throughout the motion.

* For the time taken by the particles on one of these surfaces to go once completely round, see the Note at the end of the paper.

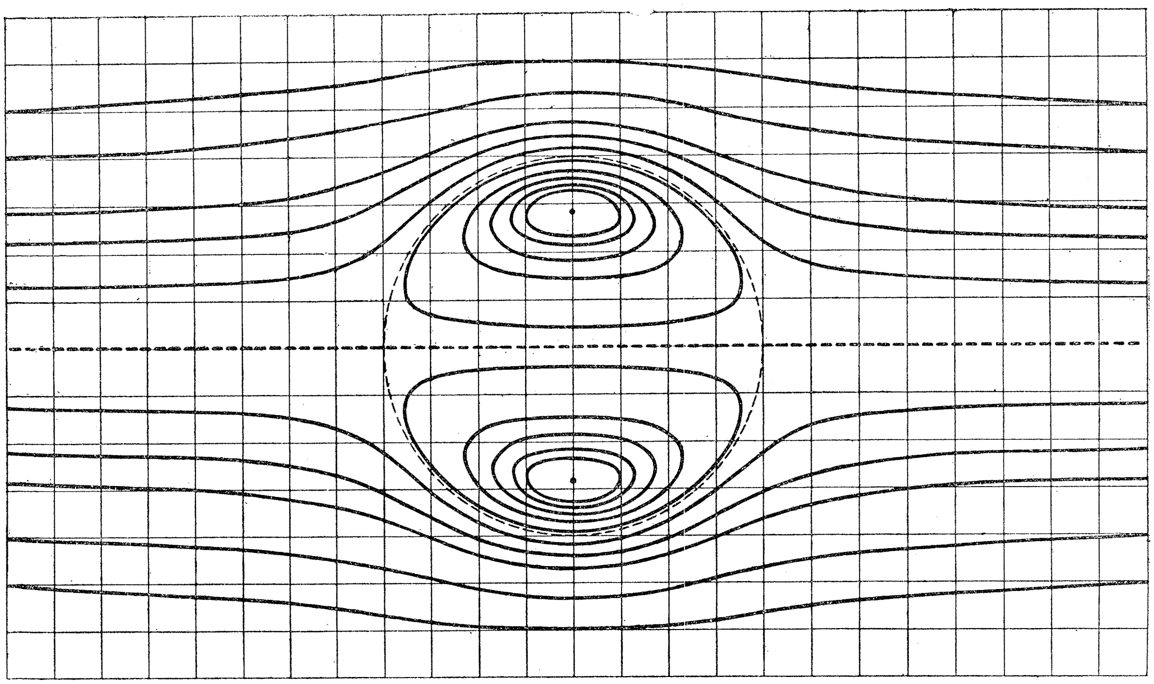
When $d^4 = \frac{1}{4} a^4$, the section of the surface, by a plane through the axis, shrinks into a point ellipse whose major axis, which is parallel to the axis of z, is double of its minor axis.

As d^4 diminishes from $a^4/4$ to 0, the surfaces increase in size until finally they become merged in the sphere $R^2 - a^2 = 0$, and the evanescent cylinder $r^2 = 0$.

Table II. gives the form of the surfaces

$$r^2 \{1 - (a/R)^3\} = d^2,$$

which are outside the sphere, and which always contain the same particles of fluid throughout the motion.



When $d^2 = 0$, the surface merges in the evanescent cylinder $r^2 = 0$, the sphere $1 - a/R = 0$, and the imaginary locus $1 + a/R + (a/R)^2 = 0$.

As d increases from 0 to ∞, the surfaces tend to become cylinders. It may be noticed that the surface $r^2 \{1 - (a/R)^3\} = d^2$ has the asymptotic cylinder $r = d$. The greatest distance of this surface from the axis is found by putting $z - Z = 0$, and, therefore, $R = r$. Hence, the greatest distance is a root of the equation

$$1 - \left(\frac{a}{r}\right)^3 = \left(\frac{d}{r}\right)^2.$$

When $r = 10\,a$ is a root of this equation.

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$$d = 10\,a\left(1 - \frac{1}{10^3}\right)^{\frac{1}{2}} = 10\,a\left(1 - \frac{1}{2\,.\,10^3}\right) \text{ nearly} = 10\,a - \frac{a}{200}.$$

This result shows how rapidly the disturbance due to the passage of the vortex sphere dies away as the distance from the axis increases.

TABLE I.—Table for calculating the surfaces of revolution $r^2(R^2 - a^2) = -d^4$.

$d^4 = \frac{a^4}{4}$	r/a	·71											
	$(z-Z)/a$	0											
$d^4 = \frac{2a^4}{9}$	r/a	·58	·63	·69	·75	·82							
	$(z-Z)/a$	0	·23	·24	·21	0							
$d^4 = \frac{a^4}{5}$	r/a	·53	·55	·6	·67	·8	·83	·85					
	$(z-Z)/a$	0	·19	·29	·32	·22	·14	0					
$d^4 = \frac{a^4}{6}$	r/a	·46	·5	·6	·64	·7	·8	·89					
	$(z-Z)/a$	0	·29	·42	·43	·41	·32	0					
$d^4 = \frac{a^4}{9}$	r/a	·36	·4	·5	·58	·7	·8	·93					
	$(z-Z)/a$	0	·38	·55	·58	·53	·43	0					
$d^4 = \frac{a^4}{81}$	r/a	·11	·13	·2	·33	·4	·5	·6	·7	·8	·9	·95	·99
	$(z-Z)/a$	0	·5	·81	·88	·87	·84	·78	·7	·58	·42	·29	0

TABLE II.—Table for calculating the surfaces of revolution $r^2\left(1 - \left(\frac{a}{R}\right)^3\right) = d^2$.

$d^2 = a^2(·1)$,	r/a	1·03	1	·9	·8	·7	·6	·5	·4	·36	·34	·33	·32
	$(z-Z)/a$	0		·27	·53	·69	·82	·94	1·08	1·33	1·6	1·92	2·28	∞
$d^2 = a^2(·3)$,	r/a	1·1	1·05	1	·9	·8	·7	·6	·57	·56	·55		
	$(z-Z)/a$	0	·37	·52	·74	·94	1·18	1·72	2·28	2·79	∞		
$d^2 = a^2(·5)$,	r/a	1·17	1·1	1	·9	·8	·75	·71					
	$(z-Z)/a$	0	·46	·77	1·04	1·46	1·94	∞					
$d^2 = a^2$,	r/a	1·325	1·3	1·2	1·1	1							
	$(z-Z)/a$	0	·36	·87	1·42	∞							
$d^2 = a^2(1·6)$,	r/a	1·5	1·4	1·3	1·26								
	$(z-Z)/a$	0	1·06	2·3	∞								

Art. 15. *Consideration of the case where the rotationally moving fluid is limited by the ellipsoid of revolution*

$$\frac{r^2}{a^2} + \frac{(z - Z)^2}{c^2} = 1.$$

In this case

$$\tau = 2\,\frac{k}{c^2}\,r\,(z - Z)$$

$$w = Z - \frac{2k}{a^2}\,(2r^2 - a^2) - \frac{2k}{c^2}\,(z - Z)^2.$$

Also

$$\frac{p}{\rho} + V = \frac{2k^2}{a^2c^2}\left(r^2 - \frac{a^2}{2}\right)^2 - \ddot{Z}\,(z - Z) - \frac{2k^2}{c^4}\,(z - Z)^4 + \frac{4k^2}{c^2}\,(z - Z)^2$$

$$+ \text{ an arbitrary function of } t.$$

Now the velocity potential due to the motion of the ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{(z - Z)^2}{c^2} = 1,$$

moving with velocity $\dot{Z}$ parallel to the axis of z, is

$$\phi = \mu\,(z - Z)\int_\epsilon^\infty \frac{du}{(a^2 + u)^{1/2}\,(b^2 + u)^{1/2}\,(c^2 + u)^{3/2}},$$

where

$$\dot{Z} = \mu\int_0^\infty \frac{du}{(a^2 + u)^{1/2}\,(b^2 + u)^{1/2}\,(c^2 + u)^{3/2}} - \frac{2\mu}{abc},$$

and ϵ is the parameter of the confocal ellipsoid through the point x, y, z. See BASSET's 'Hydrodynamics,' vol. I., Art. 147.

Then if q be the perpendicular from the centre of the ellipsoid on to a tangent plane, the velocity components *at the surface* are—

$$\frac{\partial\phi}{\partial x} = -\frac{2\mu\,(z - Z)}{abc^3}\cdot\frac{q^2x}{a^2}$$

$$\frac{\partial\phi}{\partial y} = -\frac{2\mu\,(z - Z)}{abc^3}\cdot\frac{q^2y}{b^2}$$

$$\frac{\partial\phi}{\partial z} = -\frac{2\mu\,(z - Z)}{abc^3}\cdot\frac{q^2\,(z - Z)}{c^2} + \mu\int_0^\infty \frac{du}{(a^2 + u)^{1/2}\,(b^2 + u)^{1/2}\,(c^2 + u)^{3/2}}.$$

The normal velocity at the surface is therefore

$$\frac{q\dot{Z}\,(z - Z)}{c^2},$$

and as

$$\tau \frac{qr}{a^2} + w \frac{q(z - Z)}{c^2}$$

is equal to the same expression, it is obvious that the normal velocity is continuous at the surface of the ellipsoid.

But $p/\rho + V$ is not continuous.

For

$$\frac{p}{\rho} + V + \frac{\partial \phi}{\partial t} + \tfrac{1}{2}\left(\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 + \left(\frac{\partial \phi}{\partial z}\right)^2\right) = \text{an arbitrary function of } t,$$

and since (taking $\dot{Z}$ constant),

$$\frac{\partial \phi}{\partial t} = - \dot{Z} \frac{\partial \phi}{\partial z},$$

and since, in this case, $b = a$

$$\frac{p}{\rho} + V - \mu \dot{Z} \int_0^\infty \frac{du}{(a^2 + u)(c^2 + u)^{3/2}} + \frac{2q^2 \mu \dot{Z}(z - Z)^2}{a^2 c^5}$$
$$+ \tfrac{1}{2}\left[\left\{\mu \int_0^\infty \frac{du}{(a^2 + u)(c^2 + u)^{3/2}}\right\}^2 - \frac{4q^2\mu^2(z - Z)^2}{a^2 c^5} \int_0^\infty \frac{du}{(a^2 + u)(c^2 + u)^{3/2}} + \frac{4q^2\mu^2(z - Z)^2}{a^4 c^6}\right]$$
$$= \text{an arbitrary function of } t.$$

Therefore

$$\frac{p}{\rho} + V + \frac{2q^2\mu(z - Z)^2}{a^2 c^5}\left\{\dot{Z} - \mu \int_0^\infty \frac{du}{(a^2 + u)(c^2 + u)^{3/2}} + \frac{\mu}{a^2 c}\right\} = \text{an arbitrary function of } t.$$

But

$$\dot{Z} = \mu \int_0^\infty \frac{du}{(a^2 + u)(c^2 + u)^{3/2}} - \frac{2\mu}{a^2 c},$$

therefore

$$\frac{p}{\rho} + V = \frac{2q^2\mu^2(z - Z)^2}{a^4 c^6} + \text{an arbitrary function of } t.$$

This value of $p/\rho + V$ is not continuous with the value of $p/\rho + V$ inside the ellipsoid.

Further, on returning to rectangular axes in three dimensions,

$$u = 2 \frac{k}{c^2} x (z - Z),$$

$$v = 2 \frac{k}{c^2} y (z - Z),$$

$$w = \dot{Z} - 2 \frac{k}{a^2}(2x^2 + 2y^2 - a^2) - 2 \frac{k}{c^2}(z - Z)^2.$$

Hence, if ξ, η, ζ be the components of the molecular rotation,

$$\xi = \tfrac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = -\left(\frac{4k}{a^2} + \frac{k}{c^2}\right) y,$$

$$\eta = \tfrac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) = \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) x,$$

$$\zeta = \tfrac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0.$$

Now, HELMHOLTZ's method gives the following values for u, v, w as deduced from ξ, η, ζ,

$$u = \frac{\partial P}{\partial x} + \frac{\partial N}{\partial y} - \frac{\partial M}{\partial z},$$

$$v = \frac{\partial P}{\partial y} + \frac{\partial L}{\partial z} - \frac{\partial N}{\partial x},$$

$$w = \frac{\partial P}{\partial z} + \frac{\partial M}{\partial x} - \frac{\partial L}{\partial y},$$

where

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} + \frac{\partial^2 P}{\partial z^2} = 0,$$

and L, M, N, are the potentials of $\xi/2\pi$, $\eta/2\pi$, $\zeta/2\pi$ respectively, taken throughout the rotationally moving fluid.

Hence, if the rotationally moving fluid be limited to the ellipsoid of revolution above, the values of L, M, N may be worked out completely.

For it is known that a solid ellipsoid of density, μx, gives for potential outside the ellipsoid,

$$\mu\pi a^3 bcx \int_\epsilon^\infty \left(1 - \frac{x^2}{a^2+u} - \frac{y^2}{b^2+u} - \frac{z^2}{c^2+u}\right) \frac{du}{(a^2+u)^{3/2}(b^2+u)^{1/2}(c^2+u)^{1/2}},$$

where ϵ is the positive value of λ satisfying

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} + \frac{z^2}{c^2+\lambda} = 1.$$

Inside the ellipsoid the potential has the same value if the lower limit of the integral, ϵ, be replaced by zero.

(See a paper, by Mr. DYSON, "On the Potentials of Ellipsoids," in the 'Quarterly Journal of Mathematics,' vol. 25, 1891.)

Hence, outside the ellipsoid,

$$L = -\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\frac{k}{2} a^4 cy \int_\epsilon^\infty \left(1 - \frac{r^2}{a^2+u} - \frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2 (c^2+u)^{1/2}},$$

$$M = \left(\frac{4}{a^2} + \frac{1}{c^2}\right)\frac{k}{2} a^4 cx \int_\epsilon^\infty \left(1 - \frac{r^2}{a^2+u} - \frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2 (c^2+u)^{1/2}}$$

$$N = 0.$$

Hence,

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) x (z - Z) \int_\epsilon^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}},$$

$$\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} = a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) y (z - Z) \int_\epsilon^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}},$$

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) \int_\epsilon^\infty \left(1 - \frac{2r^2}{a^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right) \frac{du}{(a^2 + u)^2 (c^2 + u)^{1/2}}$$

The values inside the ellipsoid are obtained by replacing ϵ by zero.

Outside the ellipsoid the expressions

$$\frac{\partial N}{\partial y} - \frac{\partial M}{\partial z} = \frac{\partial \phi}{\partial x}$$

$$\frac{\partial L}{\partial z} - \frac{\partial N}{\partial x} = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} = \frac{\partial \phi}{\partial z}$$

where

$$\phi = -\tfrac{1}{2} a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) (z - Z) \int_\epsilon^\infty \left(1 - \frac{r^2}{a^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right) \frac{du}{(a^2 + u)(c^2 + u)^{3/2}}$$

as may be immediately verified by differentiation.

ϕ is obviously a potential function, viz., it is what

$$-\tfrac{1}{2} a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) (z - Z) \int_\epsilon^\infty \left(1 - \frac{x^2}{a^2 + u} - \frac{y^2}{b^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right) \frac{du}{(a^2 + u)^{1/2} (b^2 + u)^{1/2} (c^2 + u)^{3/2}}$$

becomes when $a = b$.

Moreover, if k be suitably determined, it is the velocity potential for the fluid outside the ellipsoid moving with velocity $\dot{Z}$ parallel to the axis of z. (See BASSET'S "Hydrodynamics," vol. I., Art. 147.)

Inside the ellipsoid the values of $\partial N/\partial y - \partial M/\partial z$, &c., can be deduced by putting $\epsilon = 0$, and it appears that they do not give the original expressions for u, v, w.

Hence in this case the function P exists.

It is such that

$$\frac{\partial P}{\partial x} = 2 \frac{k}{c^2} x (z - Z) - a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) x (z - Z) \int_0^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}}$$

$$\frac{\partial P}{\partial y} = 2 \frac{k}{c^2} y (z - Z) - a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) y (z - Z) \int_0^\infty \frac{du}{(a^2 + u)^2 (c^2 + u)^{3/2}}$$

$$\frac{\partial P}{\partial z} = \dot{Z} - 2 \frac{k}{a^2} (2r^2 - a^2) - 2 \frac{k}{c^2} (z - Z)^2$$
$$- a^4 c \left(\frac{4k}{a^2} + \frac{k}{c^2}\right) \int_0^\infty \left(1 - \frac{2r^2}{a^2 + u} - \frac{(z - Z)^2}{c^2 + u}\right) \frac{du}{(a^2 + u)^2 (c^2 + u)^{1/2}},$$

so that

$$\frac{\partial P}{\partial r} = 2\frac{k}{c^2} r(z-Z) - a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right) r(z-Z)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}.$$

Hence

$$P = \left[\frac{k}{c^2} - \frac{k}{2} a^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}}\right](r^2(z-Z) - \tfrac{2}{3}(z-Z)^3)$$

$$+\left[\dot{Z} + 2k - a^4 c\left(\frac{4k}{a^2} + \frac{k}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}}\right](z-Z),$$

and P is a potential function, for it satisfies

$$\frac{\partial^2 P}{\partial r^2} + \frac{1}{r}\frac{\partial P}{\partial r} + \frac{\partial^2 P}{\partial z^2} = 0.$$

It appears, then, that on attempting to obtain the values of the velocity components from the molecular rotations by means of HELMHOLTZ's method, it is necessary to introduce the function P. This points to the existence of rotational motion outside the ellipsoid (as was previously remarked), P being the potential of the irrotational motion inside the ellipsoid due to the vortices outside the ellipsoid.

If P be left out of account altogether, and an attempt be made to see whether the velocity components $\partial N/\partial y - \partial M/\partial z$, $\partial L/\partial z - \partial N/\partial x$, $\partial M/\partial x - \partial L/\partial y$, which give continuous velocity at the surface of the ellipsoid, will not also give continuous pressure; then inside the ellipsoid

$$\tau = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right) r(z-Z)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}},$$

$$w = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \left(1 - \frac{2r^2}{a^2+u} - \frac{(z-Z)^2}{c^2+u}\right)\frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

or putting

$$l = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{1/2}},$$

$$m = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^3(c^2+u)^{1/2}},$$

$$n = ka^4 c\left(\frac{4}{a^2} + \frac{1}{c^2}\right)\int_0^\infty \frac{du}{(a^2+u)^2(c^2+u)^{3/2}},$$

then

$$\tau = nr(z-Z),$$
$$w = l - 2r^2 m - (z-Z)^2 n.$$

Hence the equations

$$\frac{\partial \tau}{\partial t} + \tau \frac{\partial \tau}{\partial r} + w \frac{\partial \tau}{\partial z} = - \frac{\partial}{\partial r}\left(\frac{p}{\rho} + \mathrm{V}\right),$$

$$\frac{\partial w}{\partial t} + \tau \frac{\partial w}{\partial r} + w \frac{\partial w}{\partial z} = - \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \mathrm{V}\right)$$

become

$$- nr(\dot{\mathrm{Z}} - l) - 2mnr^3 = - \frac{\partial}{\partial r}\left(\frac{p}{\rho} + \mathrm{V}\right),$$

$$- 2n(z - \mathrm{Z})(l - \dot{\mathrm{Z}}) + 2n^2(z - \mathrm{Z})^3 = - \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \mathrm{V}\right).$$

Therefore

$$\frac{p}{\rho} + \mathrm{V} = \tfrac{1}{2} mnr^4 + \tfrac{1}{2} n(\dot{\mathrm{Z}} - l) r^2 + n(l - \dot{\mathrm{Z}})(z - \mathrm{Z})^2 - \tfrac{1}{2} n^2 (z - \mathrm{Z})^4$$
$$+ \text{ an arbitrary function of } t.$$

This value of $p/\rho + \mathrm{V}$ is not continuous with the value of $p/\rho + \mathrm{V}$ for the motion outside the ellipsoid.

Summary of Results.

A. *Rotational Motion inside the Sphere* $r^2 + (z - \mathrm{Z})^2 = a^2$.

$$\left.\begin{aligned} &\text{Velocity parallel to axis of } r = 3\dot{\mathrm{Z}}r(z - \mathrm{Z})/(2a^2) \\ &\text{Velocity parallel to axis of } z = \dot{\mathrm{Z}}\{5a^2 - 3(z - \mathrm{Z})^2 - 6r^2\}/(2a^2) \end{aligned}\right\} \quad . \quad . \quad (\text{XLV.}).$$

$$\frac{p}{\rho} + \mathrm{V} = 9\dot{\mathrm{Z}}^2[(r^2 - \tfrac{1}{2}a^2)^2 - \{(z - \mathrm{Z})^2 - a^2\}^2 + a^4]/(8a^4) + \frac{\Pi}{\rho} \quad . \quad . \quad (\text{XLVI.}).$$

Current Function $\psi = 3\dot{\mathrm{Z}}r^2\{\mathrm{R}^2 - \tfrac{5}{3}a^2\}/(4a^2)$ . . . . (XLVII.).

Surfaces containing the same particles of fluid

$$3\dot{\mathrm{Z}}r^2\{\mathrm{R}^2 - a^2\}/(4a^2) = \text{const.} \quad . \quad . \quad . \quad . \quad . \quad (\text{XLVIII.}).$$

Molecular Rotation $= 15\dot{\mathrm{Z}}r/(4a^2)$ . . . . (XLIX.).

Cyclic Constant of Vortex $= 5a\dot{\mathrm{Z}}$ . . . . (L.).

B. *On the Surface of the Sphere.*

Velocity parallel to axis of $r = \frac{3}{2}\dot{Z}\sin\theta\cos\theta$ . . . . (XXXV.).

Velocity parallel to axis of $z = \dot{Z}(1 - \frac{3}{2}\sin^2\theta)$ . . . . (XXXVI.).

$$\frac{p}{\rho} + V = \frac{9}{8}\dot{Z}^2\cos^2\theta + \frac{9\dot{Z}^2}{32} + \frac{\Pi}{\rho} \quad \text{(XLIV.)}.$$

C. *Irrotational Motion outside the Sphere.*

Velocity parallel to axis of $r = 3a^3\dot{Z}r(z - Z)/(2R^5)$ . . . . (XXXII.).

Velocity parallel to axis of $z = a^3\dot{Z}\{3(z - Z)^2 - R^2\}/(2R^5)$ . . . . (XXXIII.).

$$\frac{p}{\rho} + V = \frac{1}{8}\dot{Z}^2\left[\left\{5 - 4\left(\frac{a}{R}\right)^3 - \left(\frac{a}{R}\right)^6\right\} + 3\cos^2\theta\left\{4\left(\frac{a}{R}\right)^3 - \left(\frac{a}{R}\right)^6\right\} + \frac{9}{4}\right] + \frac{\Pi}{\rho} \quad \text{(XLIII.)}.$$

Current Function $\psi = -a^3\dot{Z}r^2/(2R^3)$ . . . . (XXXVIII.).

Surfaces containing the same particles of fluid

$$\dot{Z}r^2(R^3 - a^3)/(2R^3) = \text{const.} \quad \text{(XLI.)}.$$

Velocity potential $= -a^3\dot{Z}(z - Z)/(2R^3)$ . . . . (XXXI.).

Supplementary Remarks.

The velocity potential outside the sphere is the same as that which would be produced by the distribution throughout the sphere of matter of density

$$-15\dot{Z}(z - Z)/(8\pi a^2) \quad \text{(LI.)}.$$

The potential of this distribution inside the sphere is

$$\dot{Z}(z - Z)(3R^2 - 5a^2)/(4a^2) \quad \text{(LII.)}.$$

Calling this potential U, and expressing the velocity components in CLEBSCH's form, viz.,

$$\tau = \frac{\partial \chi}{\partial r} + \lambda \frac{\partial \mu}{\partial r},$$

$$w = \frac{\partial \chi}{\partial z} + \lambda \frac{\partial \mu}{\partial z},$$

where

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \lambda = 0,$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \mu = 0,$$

$$\left(\frac{\partial}{\partial t} + \tau \frac{\partial}{\partial r} + w \frac{\partial}{\partial z}\right) \chi = -\left(\frac{p}{\rho} + \mathrm{V}\right) + \tfrac{1}{2}(\tau^2 + w^2).$$

Then

$$\lambda = 3\dot{\mathrm{Z}} r^2 \{\mathrm{R}^2 - a^2\}/(4a^2) \quad \text{(XLVIII.)},$$

$$\mu = 5 \int \frac{dr}{(\mathrm{L} + r^2 a^2 - r^4)^{1/2}} - \frac{15\mathrm{Z}}{2a^2} \quad \text{(LXX.)},$$

$$\chi = \mathrm{U} - \int \left(\frac{\Pi}{\rho} + \tfrac{2}{3}\tfrac{9}{2}\dot{\mathrm{Z}}^2\right) dt + \frac{20a^2\lambda^2}{3\dot{\mathrm{Z}}} \int \frac{dr}{r^4 (\mathrm{L} + r^2 a^2 - r^4)^{1/2}} + \text{const.} \quad \text{(LXXXIX.)},$$

where L is to be replaced by $4a^2\lambda/(3\dot{\mathrm{Z}})$ after the integrations with regard to r have been performed.

NOTE ADDED MAY 2ND.

The time taken by the particles on the surface

$$r^2(\mathrm{R}^2 - a^2) = -d^4$$

to revolve once completely round is

$$\frac{4a^2}{3\dot{\mathrm{Z}}} \int_0^{\frac{1}{2}\pi} \{\tfrac{1}{2}a^2 + \sqrt{(\tfrac{1}{4}a^4 - d^4)} - 2 \sin^2 \phi \sqrt{(\tfrac{1}{4}a^4 - d^4)}\}^{-1/2} d\phi,$$

or putting

$$\lambda = 2(\tfrac{1}{4}a^4 - d^4)^{1/2}/\{\tfrac{1}{2}a^2 + \sqrt{(\tfrac{1}{4}a^4 - d^4)}\},$$

it is

$$\frac{4a}{3\dot{\mathrm{Z}}}(2 - \lambda)^{1/2} \int_0^{\frac{1}{2}\pi} (1 - \lambda \sin^2 \phi)^{-1/2} d\phi.$$

The extreme limits of d^4 corresponding to surfaces inside the vortex sphere are $\frac{1}{4}a^4$ and 0, and as d^4 diminishes from $\frac{1}{4}a^4$ to 0, λ increases from 0 to 1.

Putting

$$F(\lambda) = (2-\lambda)^{1/2}\int_0^{\frac{1}{2}\pi}(1-\lambda\sin^2\phi)^{-1/2}\,d\phi,$$

$$F'(\lambda) = -\tfrac{1}{2}(2-\lambda)^{-1/2}\int_0^{\frac{1}{2}\pi}\cos 2\phi\,(1-\lambda\sin^2\phi)^{-3/2}\,d\phi$$

$$= \tfrac{1}{2}(2-\lambda)^{-1/2}\int_0^{\frac{1}{4}\pi}\cos 2\phi\left[(1-\lambda\cos^2\phi)^{-3/2}-(1-\lambda\sin^2\phi)^{-3/2}\right]d\phi.$$

Since $0 < \phi < \frac{1}{4}\pi$, every element of the integral is positive.

Hence $F'(\lambda)$ is positive; and, therefore, as λ increases from 0 to 1, $F(\lambda)$ increases from π to ∞.

Hence as d^4 diminishes from $\frac{1}{4}a^4$ to 0, the time of revolution increases from $4a\pi/3\dot{Z}$ to ∞.

The fact, that when $d^4 = 0$, the time is infinitely great, may be verified by finding the time along the axis of the vortex sphere from end to end, and the time along a meridian from one end of the axis to the other.

These are

$$\frac{2a^2}{3\dot{Z}}\int_{-a}^{+a}\frac{d(z-Z)}{a^2-(z-Z)^2},$$

and

$$\frac{4a}{3\dot{Z}}\int_0^{\frac{1}{2}\pi}\operatorname{cosec}\theta\,d\theta,$$

both of which are infinitely great.

This result does not constitute a difficulty, for if a particle anywhere on the axis of the sphere could reach the extremity then it would not be clear along which meridian of the sphere it should subsequently move.

If again the particles on any meridian of the sphere could reach the extremity of the axis, there would at that extremity be a collision of the particles coming in from all possible meridians.

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